DOCUMENT RESUME

ED 338 496 SE 052 348

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TITLE Reasoning and Proof in Precalculus and Discrete Mathematics.
SPONS AGENCY General Electric Foundation, Ossining, N.Y.
PUB DATE Apr 91
NOTE 33p.; Paper presented at the Annual Meeting of the American Educational Research Association (Chicago, IL, April 3-7, 1991).
PUB TYPE Reports - Research/Technical (143) -- Speeches/Conference Papers (150)

EDRS PRICE MF01/PC02 Plus Postage.
DESCRIPTORS High Schools; Mathematical Concepts; *Mathematical Logic; *Mathematics Achievement; Mathematics Curriculum; Mathematics Education; *Mathematics Tests; *Proof (Mathematics); *Secondary School Mathematics; Trigonometry
IDENTIFIERS Discrete Mathematics; Number Theory; *School Mathematics Project

ABSTRACT

Precalculus and Discrete Mathematics (PDM) is the sixth and final course in the secondary mathematics curriculum developed by the University of Chicago (Illinois) School Mathematics Project. During the 1989-90 academic year, a formative evaluation of the third field-trial edition of PDM was conducted among a volunteer sample of 9 high schools with widely varying demographics across 8 states comprising a total of 180 students. Part of the evaluation focused on that portion of student achievement associated with mathematical proof in a nongeometric context, specifically content and/or process properties of the following: even and odd integers and divisibility, proof by contradiction, trigonometric identities, and proof using the principle of mathematical induction. Results of student success on these nongeometric proofs generally settled into three nonoverlapping categories: (1) success with proofs involving trigonometric identities extended to an 80% level, a result most likely due to the fact that trigonometry had been studied immediately prior to the evaluation; (2) roughly one-third of the students were able to find a counter-example to disprove a number theory statement--this is in line with the findings from the 1989 National Assessment of Educational Progress; and (3) students showed an enormous difficulty with indirect proofs, that is, proof by contradiction and proof by induction, a finding not completely unexpected due to the considered overemphasis on form rather than substance within the mathematics curriculum. For the most part, the errors that occurred in proof achievement were consistent across all nine schools, a finding that suggests that the teacher is not a significant variable in determining proof achievement. (JJK)

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REASONING AND PROOF IN PRECALCULUS AND DISCRETE MATHEMATICS

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Paper presented at the annual meeting of the
American Educational Research Association
Chicago, IL
April 3-7, 1991

This work was partially funded by support from the General Electric Foundation to the University of Chicago School Mathematics Project.

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Reasoning and Proof in Precalculus and Discrete Mathematics

Precalculus and Discrete Mathematics (*PDM*) is the sixth and final course in the secondary mathematics curriculum developed by the University of Chicago School Mathematics Project (UCSMP). The intended audience for this course consists of students in one of two broad groups: college-bound students who have completed the previous UCSMP course entitled *Functions, Statistics, and Trigonometry with Computers*; or college-bound students who have a strong background in advanced algebra topics and considerable work with trigonometric, logarithmic, exponential, and polynomial functions [UCSMP, 1989b].

One of the major content strands in the course is reasoning and proof. Throughout the development of *PDM*, there has been an attempt to use mathematical thinking, with specific attention to formal logic and proof, as a thread to unify the themes of the course [UCSMP, 1989a]. In particular, proof is discussed in four content and/or process contexts: number theory properties, specifically properties of even and odd integers and properties of divisibility; proof by contradiction; trigonometric identitites; and proof using the principle of mathematical induction.

During the 1989-90 school year, a formative evaluation of the field trial edition was conducted. This edition represented the third version of the course materials and incorporated changes suggested by teachers who had used one or both of the previous two pilot editions. The evaluation study focused on overall achievement in precalculus and discrete mathematics, the incorporation into the classroom of automatic graphing technology, teacher and student attitudes toward the materials and content, and student achievement with proof in a non-geometric context. This paper will focus only on a portion of the proof achievement. Information about the other aspects of the study may be found in Thompson [in progress].

Methodology

Sample

In the early months of 1989, a call for participation in the evaluation study of *Precalculus and Discrete Mathematics* was placed in professional newsletters and in the project's own newsletter. Because it was unclear how much interest would exist in this course, the first few schools that applied were accepted as study participants. The result was a sample of nine schools in eight states from the east coast to the west coast. Three of the nine schools are private, two being private co-educational day schools (K-12) and one being a private secondary co-educational boarding school. The remaining six are public schools in a variety of communities; two are in mid-size cities, two are in suburbs of a major metropolitan area, and two are magnet schools in urban school districts.

At five of the sites, *PDM* is essentially a new course in the curriculum, either providing an alternative to Advanced Placement calculus or replacing a course previously offered in the non-AP track. At three sites, *PDM* is offered in the honors track to students completing an honors level second-year algebra course; these students should have been enrolled in AP calculus during the 90-91 school year. At the final site, AP calculus is not offered and *PDM* is the highest level course provided to students. A further description of the schools and the place of *PDM* in the curriculum at each site may be found in Thompson [in progress].

At these nine sites, 180 students began the school year in *Precalculus and Discrete Mathematics*. The proof test results discussed in this paper are based on a sample of 145 of these students. Of these 145, 98 (67.6%) were seniors, 41 (28.3%) were juniors, and 6 (4.1%) were sophomores. The sample included 78 (53.8%) males and 67 (46.2%) females. Most seniors were taking *PDM* as a fifth-year mathematics

course, meaning they had completed one college-preparatory mathematics course after second-year algebra. At four sites, for most students that course was the previous UCSMP course.

Instrument

To assess proof-writing ability without overtesting students at the end of the school year, a proof test was developed to be administered after completion of Chapter 6 of the text. At this point in the course, all the forms of proof, except mathematical induction, would have been covered. Thus, assessment on proof at the end of the year could be kept to a minimum. This paper focuses only on results from this mid-course proof test; the proof items on the posttest will be discussed at a later point in time.

The purpose of the proof test was to assess student ability on proofs dealing with number theory properties, trigonometric identities, the use of graphing technology to test identities, and the process and use of indirect proof. Piloting of the proof test in a non-study school using *Precalculus and Discrete Mathematics* was conducted to aid in the development of the proof instrument.

The final proof test consists of two non-overlapping forms of twelve items each. Each form contains four items dealing with number theory properties, three items dealing with trigonometric identities, one item dealing with using graphing technology to test the truth of a conjecture, one item identifying rational or irrational numbers, and three items dealing with indirect proof. Complete copies of the instrument are available from UCSMP or may be found in Thompson [in progress].

Forms were alternated when sent to the teachers so that roughly half of the students in each class took each form. The tests were administered after completion of Chapter 6 in the text, regardless of when the class reached that point. Hence, the test was administered from January through the

middle of May. Tests for reliability yield values for Cronbach α of .794 for Form 1 and .617 for Form 2.

Scoring Procedures

The proof items were graded by two experienced teachers whom I hired and trained. Holistic grading schemes for use with problem solving situations have been developed by Malone, Douglas, Kissane, and Mortock [1980] and Charles, Lester, and O'Daffer [1987] and used by Senk [1983] in scoring proofs in geometry. The system developed by Malone et al. uses a scale from 0 though 4 (3 and 4 are considered successful) and had been used successfully to grade open-ended items from other UCSMP evaluations. Hence, a decision was made to use that system as a general rubric to guide the overall scoring as well as the development of specific criteria for scoring each item. One modification to the system was to use a score of 8 to distinguish no attempt on a problem from a meaningless attempt.

A sample of papers for each item covering the scoring range was selected by this investigator and photocopied with all student identification omitted. Prior to the grading of a particular item, graders assigned scores to these samples based on the specific criteria for the item and discrepancies were discussed prior to the grading of the actual student papers. This led either to clarification of the criteria or to some minor modifications in the criteria.

For each item, both graders independently assigned a score to each student paper. When the two graders disagreed, this investigator also marked the student paper. In that case, the final score for that item was the median of the three scores. Overall, inter-rater correlations were quite high, generally .85 or better.

Results

The items requiring a proof or a disproof or an explanation of the proof process constitute a subsample of 12

items of the 24 items that appeared on one of the two forms of the proof test. The percentage of students successful on these items (scores of 3 or 4) ranged from 3% to 84% while the percentage of students who scored a 0 or an 8 ranged from 6% to 64%. The mean scores on these items ranged from 0.67 to 3.32 out of a total of 4 points.

The following three sections contain a detailed discussion of student performance with proof in the areas of trigonometric identities, number theory properties, and indirect proof. Each discussion highlights performance in that area on several items. Additional detail on these items as well as on items not discussed may be found in Thompson [in progress].

Trigonometric Identities

Overall, the best performance on proofs occurred in the context of trigonometric identities. On the four proofs dealing with trigonometry, two on each form, the percentage successful ranged from 43% to 84% while the percentage scoring 0 ranged from 4% to 22%. The percentage who made no attempt whatsoever on the problems ranged from 1% to 15%. The mean score on these items ranged from 2.04 to 3.11 on a scale of 4 points. (On the posttest, the percentage successful on trigonometric identities was 71% and 83% on a sample of 68 and 69 students, respectively.)

Figures 1 and 2 illustrate three of the trigonometric identity items and Table 1 contains the score distribution for these items.

[Insert Figures 1 and 2 and Table 1 here.]

The trigonometric identity in Figure 1 appeared in one of two formats. In the fill-in-the-blank format, the success rate was roughly 84% while it was only 69% when students had to complete the entire proof on their own. There were two common errors on the fill-in-the-blank format among those

responses still resulting in a successful score of 3: writing the Pythagorean identity ($\sin^2 x + \cos^2 x = 1$) as the fourth step in the proof rather than just using the identity; and leaving the last step as $\frac{1}{\cos^2 x}$ rather than using the definition to rewrite this quantity as $\sec^2 x$ so that it explicitly looks like the right-hand side of the proposed identity.

Almost all students who were successful on the complete proof approached the problem by first rewriting $\tan^2 x$ as $\frac{\sin^2 x}{\cos^2 x}$ and then rewriting 1 as $\frac{\cos^2 x}{\cos^2 x}$ to obtain a common denominator. After adding the two fractions, students either used the Pythagorean identity to obtain $\frac{1}{\cos^2 x}$ or rewrote $\sin^2 x$ as $1 - \cos^2 x$ and then simplified. Most students transformed only the left side of the given identity although about one-eighth transformed both sides to $\frac{1}{\cos^2 x}$.

In fact, most students who ended up with an unsuccessful proof began the proof by rewriting $\tan^2 x$. The most common error after this point was essentially assuming the equation true and using the multiplicative property of equality to multiply both sides of the conjectured identity by $\cos^2 x$.

Four students, however, approached the problem in a different way. Two began by transforming $\sec^2 x$ and then proceeding as indicated below.

$$1 + \tan^2 x = \sec^2 x$$
$$= \frac{1}{\cos^2 x}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$
$$= 1 + \frac{\sin^2 x}{\cos^2 x}$$
$$= 1 + \tan^2 x$$

[student 13, school 3(8) - score of 4]

One began with the known identity $\cos^2 x + \sin^2 x = 1$ and then transformed this identity into the given statement. One other student began by rewriting $\tan^2 x$ but then proceeded as follows.

$$1 + \tan^2 x =$$

$$1 + \frac{\sin^2 x}{\cos^2 x} =$$

$$1 + \frac{1 - \cos^2 x}{\cos^2 x} =$$

$$1 + \frac{1}{\cos^2 x} - 1 =$$

$$\frac{1}{\cos^2 x} =$$

$$\sec^2 x = \sec^2 x$$

[student 14, school 7 – score of 4]

On the identity in Figure 2, the main error that occurred was the assumption by 12 students (16.7%) that the given statement was true without providing any proof. This error is likely a result of the fact that this identity is proved in the student text. Although the same can be said of the identity previously discussed, the current identity is one that is used quite frequently in problems and other simplifications. Hence, it is more likely that students recall this statement as an identity that they have already proved. This may explain why this item had the lowest success rate of all the trigonometric identities on either the proof test or the posttest.

Number Theory

Students were moderately successful with the four number theory proofs. On three of the four items, the success rate was roughly 33% while on the fourth item it was 47%. The

percentage of students who received a score of 0 ranged from 6% to 34%. The percentage who made no attempt on the problems, resulting in a score of 8, ranged from 0% to 6%. The mean scores on the items ranged from 1.51 to 2.29 out of 4 points.

Figures 3 through 5 illustrate the given number theory items and Table 2 contains the distribution of scores on the items.

[Insert Table 2 and Figures 3 - 5 here.]

The number theory proof shown in Figure 3 appeared in one of the two given forms. Close to half of the students correctly dealt with this problem in a fill-in-the blank format while only about one-third of the students could deal with the problem when required to complete the entire proof on their own.

Of the 14 students who were successful on the fill-in-the-blank proof with a score of 3, the most common error, made by 7 of the 14, was the failure to complete the final statement of the proof correctly:

Because _____ *is an integer, m - n is an odd integer.*

Five of the seven completed the blank with $2(r - s) + 1$, one completed the blank with $(r - s) + 1$, and one completed the blank with $2(r - s)$. These students indicate confusion over what quantity must be an integer in order for the form $2k + 1$ to represent an odd integer.

Of the ten students who received a score of 3 (successful) when required to complete the entire proof on their own, seven failed to indicate that k and j are integers when writing $m = 2k + 1$ and $n = 2j$ and 2 failed to indicate that $k - j$ is an integer before concluding that $m - n = 2(k - j) + 1$ is an odd integer. A typical response follows.

∃ integers K and L such that m = 2K + 1 by def. of odd int. and n = 2L by def. of even int. Therefor [sic]

$$m - n = (2K + 1) - 2L \quad \text{by substitution}$$
$$2K + 1 - 2L = \quad \text{by mathematics}$$
$$2K - 2L + 1 \quad \text{identity}$$
$$2(K - L) + 1$$

Since 2(K - L) is even because of definition of even integers therefore 2(K - L) + 1 is odd because of def. of odd integers there [sic] m - n is odd.
[student 14, school 4(4) – score of 3]

In both formats, the most common error, resulting in an unsuccessful score, was the representation of both m and n in terms of the same integer, ie, m = 2k + 1 and n = 2k for some integer k. On the fill-in-the-blank proof, students were provided clues to complete the representation of m and n. Nevertheless, 16 students (22.2%) represented m and n using the correct form for odd and even integers, respectively, but using the same letter to represent an integer (such as m = 2r + 1 and n = 2r). In fact, three of these 16 students represented m and n in terms of one of the given integers, namely m = 2n + 1 and n = 2n. Another six of these 16 totally ignored the given clues and wrote m = 2k + 1 and n = 2k, the exact representation used in the text definitions.

When required to complete the entire proof on their own, the number of students who made this error is greater than the total number of students who were successful on the item. Twenty-six students (35.6%) used the same letter to represent m and n and then essentially completed the proof in an appropriate manner consistent with the error; all but one of these students received a score of 2. Another 6 students (8.2%) used the same letter to specify m and n and then got lost in completing the proof. Two typical responses follow.

Suppose m is odd, then m = 2k + 1 for any integer k. Suppose n is even, then n = 2k for any integer k. m - n = 2k + 1 - 2k = 1 ⇒ one is odd therefore m - n is odd.
[student 4, school 1(2) – score of 2]

Suppose m is any odd integer and n is any even integer. Let $m = 2k + 1$ and $n = 2k$. Then,

$$m - n = 2k + 1 + 2k$$
$$= 4k + 1$$

Therefore, $m - n$ is an odd integer for any integer k.
[student 11, school 1(1) - score of 1]

Of the thirty-two students on the complete proof who represented m and n in terms of the same letter, all but two represented m and n in terms of k, a definite tie to the form of the definition as stated in their text. This error occurred despite a text discussion about the error inherent in representing two integers in the same problem, such as m and n, in terms of the same dummy variable [Peressini et al., 1989, pp. 40-43]. There are also problems in the exercise sets in which this error is made in a "proof" and students are expected to identify the error. However, it would appear that this concept has not been understood or mastered by these students.

The items in Figures 4 and 5 pose a special problem for students as there is nothing inherent in the directions to the items to clue students on how to proceed. That is, students do not have prior knowledge as to whether the given statements are true or whether they are false. Hence, students must determine what is necessary to either prove or disprove the statements. Consequently, the responses provide some clue as to how students approach such a problem. Some similar errors occurred on both items, although the item in Figure 4 is true and the item in Figure 5 is false. Roughly one-third of the students were successful on these two items.

On the item appearing in Figure 4, the divisibility of a product, most of the successful responses received a score of 3 rather than a score of 4. These twenty students began a correct proof by using the definition of factor to specify $b = a \cdot k$ and $c = a \cdot j$. However, they generally made one of three errors: they failed to indicate that k and j are integers; or they failed to indicate that $a \cdot k \cdot j$ is an

integer after obtaining $b \cdot c = a(akj)$; or they multiplied $b \cdot c$ but obtained $bc = akj$ rather than $bc = a^2kj$. Below are typical student responses with these errors.

> Suppose that ∃ integers a, b, and c such that a is a factor of b and a is a factor of c. ∴ ∃ exists a factor s such that $b = as$ and a factor r such that $c = ar$. Then, $b \cdot c = ar \cdot as = a^2(rs)$. ∴ a is a factor of $b \cdot c$.
> [student 17, school 4(6) – score of 3]

> Suppose a is a factor of b and a is a factor of c. Then by definition of factors where m and n are integers, $b = a \cdot m$ $c = a \cdot n$. Therefore, $b \cdot c$ can be written as $(a \cdot m)(a \cdot n)$ which equals $a(mn)$ making a a factor because m and n are integers.
> [student 16, school 6 – score of 3]

Another seven students (9.7%) recognized the statement as true and began a correct proof by using the definition of factor to specify $b = a \cdot k$ and $c = a \cdot j$. However, these students made several errors after this point, or failed to complete the proof, or got lost in the completion of the proof, all resulting in an unsuccessful score of 2. Typical of these responses is the following.

> Spse a, b and c are all integers
> by def. of a factor a is a factor of b, and a is a factor of c
> $ax = b$
> $ay = c$
> $a = \frac{b}{x}$
> $a = \frac{c}{y}$
> $\frac{b}{x} = \frac{c}{y}$
> $by = xc$
> [student 3, school 8 – score of 2]

Among students who did not begin a correct proof, three types of conceptual errors appeared frequently. By

far, the most common of these was an attempt by thirteen students (18.1%) to prove this statement true by using specific examples, although one of these students used examples that were incorrect. These students all received a score of 1. A typical student response is below.

a = 4
b = 8
c = 12

4 is a factor of 8
4 is a factor of 12
4 is a factor of 8 · 12
∴ 4 is a factor of 96.
[student 11, school 3(8) – score of 1]

Clearly, these students failed to distinguish between the use of a counterexample to disprove a statement and the inability of specific examples to prove a statement.

The other conceptual errors were not as frequent. Six students (8.3%) tried to prove the statement true by using the law of transitivity; these students would appear to have misread the statement and/or misunderstood the law of transitivity. Six other students specified b and c in terms of the same letter, an error consistent with that made in the item in Figure 3. Of these six, two appeared to understand the meaning of the term *factor* as they specified b and c as b = ak and c = ak; four attempted to blend a general proof with finding a specific example that works as they specified a, b and c as a = 2k, b = 4k and c = 6k or as a = 2k, b = 2k, and c = 2k.

The item in Figure 5 also requires students to make a decision about whether or not a given statement is true or false. Slightly more than one-third of the students were successful on this item and provided an appropriate counterexample. The students who received a score of 3 on this item found appropriate values for a counterexample but failed to make an explicit reference to the fact that $\frac{a}{b}$ was not an integer or that a was not divisible by b.

Consequently, the reader was left to infer that the examples provided a disproof of the claim. The differences in the responses receiving a score of 3 and those receiving a score of 4 are illustrated in the student responses below.

> Suppose that ∀ integers a and b, a^2 is divisible by b. Then suppose that a is divisible by b. Now let a = 10, and b = 25. Thus,
>
> $\frac{(10)^2}{25} = 4$, so a^2 is divisible by b.
>
> but, $\frac{10}{25} = \frac{2}{5}$, so a isn't divisible by b.
>
> [student 10, school 5 – score of 4]

> Disprove by counterexample
> Let a = 3
> Let b = 9
> $a^2 = 3^2$
> $a^2 = 9$
>
> $\frac{a^2}{b} = \frac{9}{9} = 1$
>
> $\frac{a}{b} = \frac{3}{9} = \frac{1}{3}$
>
> [student 4, school 5 – score of 3]

Among the responses receiving scores of 0 or 1, three major errors appeared. Five students (6.8%) recognized that the given statement is false but chose incorrect values for use as a counterexample. A typical response is below.

> Let a = 3
> b = 4
>
> $\frac{3^2}{4} = \frac{9}{4} = 2.25$ $\frac{3}{4} = .75$
>
> [student 12, school 3(8) – score of 1]

A second major error among the unsuccessful papers was an attempt by seven students (9.6%) to prove the statement true in general. The work indicates that these students

quickly became lost as illustrated below.

> For integer k; $a^2 = bk$ therefore;
> $a = \sqrt{bk}$ showing that b and k are factors of a
> ∴ a is divisible by b
> [student 10, school 3(8) – score of 0]

The third major error among the unsuccessful papers was an attempt by six students (8.2%) to prove the statement true by finding specific numerical examples that work. A typical response is below.

> true
> many examples
> $a^2 = 16$ $b = 2$ $\frac{16}{2} = 8$ $\frac{4}{2} = 2$
>
> $a^2 = 25$ $b = 5$ $\frac{25}{5} = 5$ $\frac{5}{5} = 1$
>
> $a^2 = 81$ $b = 3$ $\frac{81}{3} = 27$ $\frac{9}{3} = 3$
>
> The statement is true.
> [student 6, school 7 – score of 1]

This technique of proving a statement true by finding examples that satisfy the statement was also applied to the item in Figure 4. These students indicate a failure to understand the difference between the use of a counterexample to disprove a statement and the inability of specific examples to prove a statement. It appears that these students lack an understanding of the meaning of a proof.

Indirect Proof

In general, students did rather poorly on the items dealing with indirect proof. When asked to explain the process of proof by contradiction, about one-third of the students were able to do so. However, only about 16% of the students were able to write and then prove a statement that would be proved using proof by contradiction. When given an actual proof to complete, only 3% were able to successfully

do so. Figures 6-8 contain three of the indirect proof items and Table 3 illustrates the range of scores.

[insert Table 3 and Figures 6-8 here.]

Although only one-third of the students could describe the process used with an indirect argument, fifty-five students (75.3%) indicated that proof by contradiction begins with the assumption of the negation, often called opposite, of the statement to be proved. Although students seem to understand how an indirect proof begins, there is a wide range in the levels of understanding of the actual process of indirect proof.

To receive a score of 4, the response should have included the following components: the need to assume the negation of the given statement, the need to obtain a contradiction, and an understanding that the contradiction means the assumption is false and the original statement is therefore true. Ten of the fourteen responses receiving a score of 4 contained all of these elements. The other four responses failed to explicitly indicate that the contradiction means the assumption is false and the original statement is true; instead, they concluded that the original statement is true once a contradiction is obtained. These four responses should likely have received a score of 3. Two typical responses are repeated below to emphasize the difference in the sophistication of the responses.

> First, suppose that the opposite of what you're trying to prove is true. Then, proceed with the proof until you reach a contradiction. The contradiction will then prove the statement false, and the original statement true.
> [student 10, school 5 – score of 4]

1. Assume the opposite of that which you wish to prove.
2. go [sic] about the proof trying to prove this opposite.
3. eventually [sic] a contradiction will be made, thus proving the original statement true.

[student 8, school 3(1) – score of 4 (error)]

While the second response certainly suggests that the student understands the process of indirect argument, the response is not as complete or as sophisticated as the first one. It is more in line with the following responses receiving a score of 3.

> You take the statement to be proved, and assume the opposite. You then work out a proof, using the new statement until you obtain a contradiction. This contradiction proves the original statement true.
> [student 5, school 3(1) – score of 3]

> The first step is to assume the opposite of the supposition is true. Follow the necessary steps to prove it until you come across the contradiction. Once you have found the problem, you can accept the original statement as being true.
> [student 11, school 1(1) – score of 3]

Six of the responses receiving a score of 3 should likely have received a score of 2. This would adjust the overall success rate on this item to 27%.

The responses receiving a score of 2 generally failed to make any explicit reference to obtaining a contradiction in the course of the proof. Rather, these responses often referred to stating the negation and then proving it wrong or proving it false or not being able to prove it; the result is that the original statement is true. An instance of this response is below.

1. Assume that the opposite of the statement to be proven is true.
2. Go about proving that new statement from 1.
3. Since the opposite of the original statement is false, the original statement is therefore true.

[student 13, school 5 - score of 2]

There is no indication as to how the negation of the original statement is known to be false.

One misconception appeared in several responses to this item. Nine students (12.3%) implied that the original statement could be either true or false. In attempting the proof with the negation of the statement, one could find the negation true, implying that the original statement is false, or one could find the negation false, implying that the original statement is true. These responses, illustrated below, generally received a score of 1.

> Suppose not n. Add information treating the quantity as if it were negated. Then try to prove its negation and if you can its [sic] false, if not its [sic] true.
> [student 6, school 4(6) - score of 1]

One other error is worth consideration here in light of errors made in the proofs of other items on one of the two forms of the proof test. Six students (8.2%) wrote that proof by contradiction meant finding examples: five that it meant finding counterexamples and one that it meant picking examples to see if they work. It appears that these students view *contradiction* as synonomous with *false*, thus implying that a counterexample is needed. It would appear that these students lack a solid understanding of the nature of proof as well as a realization that there are many different techniques that may be used to generate a valid proof.

After being asked to write about the process of indirect proof, students were asked to write and then prove a statement that could be proved by using proof by contradiction (Figure 7). Roughly sixteen percent of the

students were successful on this item. Over half of the unsuccessful responses resulted in a score of 0.

Three types of statements dominated the responses. Twenty-one students (28.8%) wrote and attempted a proof that $\sqrt{n}$ is irrational for some specific value of n that is not a perfect square. In fact, of the twelve successful responses, nine involved such a statement. Of these 21 responses, nine were successful and twelve were not. Those receiving a score of 2 generally specified $\sqrt{n} = \frac{a}{b}$ for integers a and b ($b \neq 0$) in lowest terms but got lost after squaring both sides; those receiving a score of 1 made no progress after writing the statement.

A second major category of response, provided by 23 students (31.5%) and receiving a score of 0, involved writing a statement that would naturally be proved using a direct argument. The most common of these statements, occuring 15 times, involved some property of even and odd integers, such as the product of an even integer and an odd integer is an even integer.

The third major category of response, provided by 9 students (12.3%), involved the writing of a statement that is false, such as *all primes are odd* or *35 is even* or *all people are men*. This response is strange and may indicate a confusion between the desire to obtain a contradiction as a necessary ingredient of an indirect proof and the disproof of a false statement through examples that satisfy the hypothesis of the statement but not its conclusion.

The item in Figure 8 requires students to complete a proof by contradiction of a given statement rather than create a statement on their own. Only two of the seventy-two students who were given this item were successful, both with scores of 3. Both of the successful responses are below.

If a is rational and b is irrational then a + b is rational. a can be expressed as some number $\frac{x}{z}$ if a + b is rational then it can be expressed as $\frac{t}{v}$ $\frac{t}{v} - \frac{x}{z} = b$ If $\frac{t}{v}$ is rational and $\frac{x}{z}$ is rational then b must be rational – closure under rational numbers. therefore the assumption is false and a + b is irrational.
[student 9, school 5 – score of 3]

assume: a is a rational number and b is an irrational number and a + b is a rational number by def. of a rational number, for some integers r and s with no common factors, $a = \frac{r}{s}$

$a + b = \frac{r}{s} + b$

$= \frac{r + b \cdot s}{s}$ since $b \cdot s$ is not an integer $(r + b \cdot s)$ is not an integer

since $(r + b \cdot s)$ is not an integer $\frac{r + b \cdot s}{s}$ is irrational
this contradicts my assumption
∴ a + b is irrational
[student 2, school 3(8) – score of 3]

One of the major errors, generated by 11 students (15.2%) was the assumption of the statement to be proved. Students may have expressed a, a rational number, as a quotient of integers. But the students then added a and b, an irrational number, assuming that the sum is irrational. A range of typical responses, scoring 1 or 2, is below.

a is a rational number and b is an irrational number and a + b is not irrational
for any int. p and q, by def. of a rational #,
$a = \frac{p}{q}$

b is not expressable by a fraction
$a + b = \frac{p}{q} + b$
Since b is can not be expressed [sic] as a

fraction, the sum of a & b cannot be expressed as a fraction.
∴ a + b is irrational by def. of a rational #
contradiction
∴ assumption is false & the original statement is true
[student 6, school 5 – score of 2]

a is a rational number
b is an irrational number
and a + b is rational
b is not ending and nonrepeating behind the decimal .48923... $a = \frac{c}{d}$

$\frac{c}{d} + .\#\#\#\# = \frac{c}{d}.\#\#\#\# \neq$ rational .

∴ a + b is irrational
[student 3, school 2 – score of 1]

Suppose a is a rational # and b is irrational
then a + b is a rational number.
a + b is irrational because the sum of a rational and irrational number is irrational

Our statement is proven false
∴ a + b is an irrational number
[student 7, school 3(1) – score of 1]

In the discussion of the errors made on other items, it was noted that a number of students attempted to prove the given statements true with specific examples. The same error appeared on this item as well. Nine students (12.5%) provided a specific example to illustrate the theorem. Two typical responses follow.

Assume a is an [sic] rational number and b is an irrational number, then a + b is a rational number.
counterexample: Suppose a = 2 and b = $\sqrt{2}$,
$2 + \sqrt{2} = 2 + 1.414213562 \ldots$
$= 3.414213562 \ldots$
which cannot be written as a fraction and is therefore irrational
[student 13, school 3(8) – score of 1]

> Suppose: If a is rational and b is irrational then a + b is rational #.
> Ex is if a is 9 and b is $\sqrt{11}$, by adding you get $9 + \sqrt{11}$. Then # is irrational so original statement is true.
> [student 17, school 1(2) – score of 1]

Six additional students (8.3%) attempted to generate an example to illustrate the theorem by choosing to let the irrational number b equal some multiple of i. The sum of a and b was then found to be irrational. All but one of these students had misclassified i as irrational on another item. However, in five of the six cases it is not clear whether i is the imaginary unit or whether i is simply being used to represent an irrational number; in only one case is it clear that i is the imaginary unit as the student refers to a + bi as a complex number. The use of i for an irrational number may be due to a misconception arising from the use of I to represent an irrational number in the student text.

Number of Proofs Completed

Including the fill-in-the-blank number theory proof, Form 1 of the proof test contains 5 proof items. Including the fill-in-the-blank trigonometric identity and the explanation of the process of indirect proof, Form 2 contains 7 proof items. Table 4 summarizes the number of proofs correct on each of the two forms.

As indicated in that table, two students successfully completed all proofs on Form 1. In fact, 22% of the students completed at least four of the five proofs (80% or better) and 40% completed at least three of the five proofs (60% or better). Almost 40% were able to complete no more than 1 proof; in all but two cases, students who completed only one proof completed one of the two trigonometric identities. It should be remembered that this form contains the one indirect

proof which was correctly answered by only two students.

The results in Table 4 would suggest that the proofs on Form 2 were more difficult. Two of the proofs on this form had very low success rates; one of the two is the item in Figure 7 and the other is an item not discussed here that had a success rate of 4%. No student on this form completed all the proofs correctly; in fact, only 1 student completed 6 of the 7. Only 23% of the students completed at least four of the seven proofs (57% or better). Roughly one-fourth of the students completed no more than 1 proof; in all but five cases that one proof was the fill-in-the-blank trigonometric identity.

Discussion

The present study considers proof achievement as one aspect of an evaluation of a new course in precalculus and discrete mathematics. In contrast to many studies of proof which consider proof achievement in only one content area, the present study considers proof-writing achievement in a variety of content and/or process areas. Hence, it provides an opportunity to study successes and difficulties that may cut across a variety of proof situations.

Success on the non-geometric proofs discussed in this paper generally fall in three non-overlapping groups. Trigonometric identities were by far the easiest proofs for students to complete; with one exception, success rates on these items were over 55%. The fact that trigonometric identities were studied in the last chapter prior to administration of the proof test may have accounted for the high success. If that were so, then one would expect the success on trigonometric items on the posttest to be much lower. However, success on those items ranged from 70% to 80% [Thompson, in progress] and are within the same range as success rates on the proof test. It is more likely that such proofs are easier for students because there is a clearer

indication of how to proceed. One must use a basic definition and transform one side of the proposed identity until it agrees with the other side.

These clues or strategies about how to begin a proof are less apparent with number theory proofs or indirect proofs. With the exception of the fill-in-the-blank proof, success on the number theory items hovered around 33%. Similar results were found on the posttest. The main difficulty with a proof here is using a definition of factor or even/odd integer to represent a given number in a form that is beneficial to complete a proof. If students fail to understand the somewhat arbitrary nature of that representation, then students may fail to distinguish different numbers in different forms. This was the problem here. Students appeared bound to a specific representation as provided in their text; that representation was insufficient to accomplish the task.

Roughly one-third of the students were able to find a counterexample to disprove a statement in a number theory context. These results are in line with those on the fourth National Assessment of Educational Progress [Lindquist, 1989] in which 39% of the 11th grade students correctly found a counterexample in an algebra setting and 31% in a geometry setting. In both of those cases, students were clearly directed to find a counterexample. In the number theory context discussed here, students had no prior knowledge as to whether a counterexample or a proof was required. Hence, the results here would suggest that students have an understanding of the means by which to disprove a statement.

The enormous difficulty that students had with indirect proof is not completely unexpected. Indirect argument is a concept that requires repeated use and discussion in order for students to achieve mastery. Although the exact items are not reported in Lindquist [1989], the fourth NAEP found fewer than half of the eleventh graders able to deal with

indirect proof in a straightforward context; Williams [1979] found less than 20% of eleventh graders in a college-preparatory course able to deal with indirect proof. The difficulty encountered in the current study was likely compounded by the context of rational and irrational numbers. Unfortunately, no indirect proof items were given in another context.

When asked to generate a statement requiring an indirect argument, most of the self-generated statements tended to be either a proof of the irrationality of a square root or an attempted proof of statements typically completed using a direct approach. This would suggest a shallow understanding of the power of indirect argument and the types of statements for which it is the appropriate proof technique. The fact that students can describe, but not execute, the process of indirect proof suggests a possible overemphasis on form rather than substance. That is, students may fixate on the format of an indirect agrument rather than concentrating on the underlying reasoning.

Of real concern is a misconception that occurred in the context of number theory as well as indirect proof. Quite a few students "proved" a statement by supplying specific examples that satisfy the given conditions. This suggests a failure to understand the nature of proof as well as confusion over the use of examples to disprove a statement but the inability of examples to prove a statement. In the given items, there were no clues as to whether a proof was needed or whether a counterexample was needed to disprove the statement. Had such clues been available, the same types of errors may or may not have been present. Unfortunately, the proof test did not contain similar items for which students knew how to begin or what final result was expected.

However, both Bell [1976] and Williams [1979] have found students willing to accept a few specific cases as reason to jump to a generalization of a statement. This suggests that

the underlying knowledge of the nature of a mathematical proof may not be covered in sufficient depth in the curriculum. As Williams suggests, students fail to see a need to prove a proposition which they consider to be obvious.

Implications for Further Research

For the most part, the errors that occurred in proof achievement were consistent across the nine schools. This suggests that the teacher was not a significant variable in determining proof achievement. However, there was essentially no classroom observation during the time that students were introduced to the various proof contexts. Combining such observation with assessment of proof-writing ability might clarify how misconceptions arise and are handled in the classroom. The types of student questions when proofs are introduced may lead researchers to a better understanding of how students internalize such concepts.

Further, although a pretest was administered, there was no assessment of proof-writing ability at the beginning of the course. The students in *PDM* are already better mathematics students because they have continued beyond second-year algebra. Are those successful with proof here those who were the most successful with proof in earlier courses? What relationships, if any, exist between proof-writing ability in geometry and proof-writing ability in these contexts?

The use of specific cases to justify a statement suggests that we need to take a long look at the curriculum. It is not uncommon for students to be introduced to theorems whose proofs may or may not be accessible in a given course. Is it possible that in explaining what the theorem means, often with specific cases, that we are laying the groundwork for the misconception that such instances prove the theorem true? It would be instructive to assess students, possibly

through interviews, after such discussions in class.

In summary, this research provides a look at a group of students in a wide variety of proof situations. Those situations result in vast differences in achievement. More detailed study of student understanding is needed to determine why such differences occur. In addition, more work needs to be done in assessing how students decide on an approach when confronted with a statement whose truth is unknown.

Figure 1: Trigonometric Identity $1 + \tan^2 x = \sec^2 x$

Form 1 (Item 11)

Prove the following trigonometric identity:

For all real numbers for which both sides are defined,

$$1 + \tan^2 x = \sec^2 x.$$

Form 2 (Item 11)

Supply the missing steps in the proof of the following identity.

For all real numbers for which both sides are defined,

$$1 + \tan^2 x = \sec^2 x.$$

Proof:

left-hand side = $1 + \tan^2 x$

$= 1 +$ _____ definition of tangent

$=$ _____ $+$ _____ forming a common denominator

$=$ _____ adding fractions with a common denominator

$=$ _____ Pythagorean identity

$=$ _____ definition of secant

= right-hand side

Therefore, by the transitive property of equality,

_____.

Figure 2: Trigonometric Identity for sin 2x.

Prove or disprove the following conjecture.

$\sin 2x = 2 \sin x \cos x.$

Figure 3: The difference of an odd and even integer

Form 1 (Item 2)

Supply the missing steps in the proof of the following theorem:

If m is any odd integer and n is any even integer, then m - n is an odd integer.

Proof:
Suppose _____. Then by the definition of odd and even integers, there exists an integer r such that m = _____ and there exists an integer s such that n = _____. Then m - n = _____ = 2 · _____ + 1. Because _____ is an integer, m - n is an odd integer.

Form 2 (Item 3)

Prove the following theorem:

If m is an odd integer and n is an even integer, then m - n is an odd integer.

Figure 4: Divisibility of a Product (Item 4, Form 1)

Is the statement below true or false? If true, prove that it is true. If false, prove that it is false.

For all integers a, b, and c, if a is a factor of b and a is a factor of c, then a is a factor of b · c.

Figure 5: Divisibility Statement Requiring a Counterexample (Item 4, Form 2)

Prove or disprove the following conjecture.

For all integers a and b, if a^2 is divisible by b then a is divisible by b.

Figure 6: Process of Indirect Proof (Item 6, Form 2)

Explain the process used to write a proof by contradiction.

Figure 7: Generating a Statement Requiring Indirect Proof (Item 7, Form 2)

> Write a statement that can be proved by using proof by contradiction. Then write the proof.

Figure 8: Indirect Proof (Item 8, Form 1)

> Use a proof by contradiction to prove the following statement is true:
>
> *If a is a rational number and b is an irrational number, then a + b is an irrational number.*

Table 1: Score Distribution for Trigonometric Identities

Item	n	mean score	percent receiving each score 4	3	2	1	0	8
2-10	73	2.04	43	0	6	23	14	15
1-11	72	2.89	54	15	4	18	4	4
2-11	73	3.32	64	19	6	6	4	1

Table 2: Score Distribution for Number Theory Items

Item	n	mean score	percent receiving each score 4	3	2	1	0	8
1-2	72	2.29	28	19	17	26	10	0
2-3	73	2.21	19	14	47	10	11	0
1-4	72	1.76	6	28	14	43	6	4
2-4	73	1.51	22	12	1	23	34	7

Table 3: Score Distribution for Indirect Proof Items

Item	n	mean score	percent receiving each score 4	3	2	1	0	8
1-8	72	0.67	0	3	6	47	38	7
2-6	73	1.92	19	16	22	22	14	7
2-7	73	0.92	12	4	11	8	52	12

Table 4: Distribution of Number of Proofs Correct

Number of Proofs Correct	Form 1		Form 2	
	n	%	n	%
0	12	17	5	7
1	16	22	13	18
2	15	21	16	22
3	13	18	22	30
4	14	19	12	16
5	2	3	4	5
6	na		1	1
7	na		0	0
Total	72	100	73	99

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